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The Brun–Hooley Sieve

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1. INTRODUCTION

The object of this note is to give an alternative and, we think, simpler account of the Brun–Hooley sieve (see [Ho]) and to derive a general theorem that is in a form ready for numerous applications. We shall put forward also a "dual" form of Hooley's method that probably has relevance to the multi-dimensional vector sieve of Brüdern and Fouvry ([BF1, BF2]).

Let $\mathscr{A}$ denote a finite integer sequence of about X elements and let $\mathscr{P}$ be a finite set of primes. Writing $P=\prod_{p\in\mathscr{P}} p$ and (a, b) for the highest common factor of a and b, our objective is to estimate the counting number

$$S(\mathscr{A}, \mathscr{P}) := |\{a\in\mathscr{A} : (a, P)=1\}|.$$

The indicator function of the sub-set of $\mathscr{A}$ whose cardinality is $S(\mathscr{A}, \mathscr{P})$ is

$$\sum_{d\mid(a, P)} \mu(d), \qquad a\in\mathscr{A};$$

and it is well known from Brun's "pure" sieve (equivalently, Bonferroni's inequalities) that if $\nu(d)$ denotes the number of prime divisors of d and k is an even natural number, then

$$\sum_{d\mid(a, P)} \mu(d) \leqslant \sum_{\substack{d\mid(a, P)\\ \nu(d)\leqslant k}} \mu(d). \tag{1}$$

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Now let

$$\mathscr{P} = \bigcup_{j=1}^{r} \mathscr{P}_j$$

be a partition of $\mathscr{P}$ (so that $\mathscr{P}_i \cap \mathscr{P}_j = \phi$ if $i \neq j$) and write $P_j = \prod_{p \in \mathscr{P}_j} p$. Then, following Hooley,

$$\begin{aligned} \sum_{d \mid (a, P)} \mu(d) &= \prod_{j=1}^{r} \sum_{d \mid (a, P_j)} \mu(d) \\ &\leqslant \prod_{j=1}^{r} \sum_{\substack{d \mid (a, P_j) \\ \nu(d) \leqslant k_j}} \mu(d). \end{aligned} \tag{2}$$

for any choice of r positive even integers $k_1, ..., k_r$; and consequently

$$\begin{aligned} S(\mathscr{A}, \mathscr{P}) &= \sum_{a \in \mathscr{A}} \sum_{d \mid (a, P)} \mu(d) \\ &\leqslant \sum_{\substack{d_1, ..., d_r \\ d_j \mid P_j,\, \nu(d_j) \leqslant k_j}} \mu(d_1) \cdots \mu(d_r) \, |\{a \in \mathscr{A} : d_1 \cdots d_r \mid a\}|. \end{aligned} \tag{3}$$

In Brun's "pure" sieve the inequality in (1) is reversed if k is odd, but for $r \geqslant 2$ there is no such simple counterpart to (2). To find a lower bound for $S(\mathscr{A}, \mathscr{P})$ Hooley derives an identity that is rather complicated to prove and to state, but we can reach much the same conclusion via the following simple inequality:

LEMMA 1. *Suppose that* $0 \leqslant x_j \leqslant y_j (j = 1, ..., r)$. *Then*

$$x_1 \cdots x_r \geqslant y_1 \cdots y_r - \sum_{\ell=1}^{r} (y_\ell - x_\ell) \prod_{\substack{j=1 \\ j \neq \ell}}^{r} y_j. \tag{4}$$

Proof. The inequality holds (with equality) when $r = 1$, and follows by induction on r from

$$\begin{aligned} y_1 \cdots y_r - x_1 \cdots x_r &= (y_1 \cdots y_{r-1} - x_1 \cdots x_{r-1})\, y_r + x_1 \cdots x_{r-1}(y_r - x_r) \\ &\leqslant (y_1 \cdots y_{r-1} - x_1 \cdots x_{r-1})\, y_r + y_1 \cdots y_{r-1}(y_r - x_r). \quad \blacksquare \end{aligned}$$

We apply the inequality with

$$x_j = \sum_{d \mid (a, P_j)} \mu(d), \qquad y_j = \sum_{\substack{d \mid (a, P_j) \\ \nu(d) \leqslant k_j}} \mu(d) \qquad (j = 1, ..., r);$$

from Brun's "pure" sieve (see for example, [HR, Chapter 2, (2.4)])

$$0 \leqslant y_\ell - x_\ell \leqslant \sum_{\substack{d \mid (a, P_\ell) \\ \nu(d) = k_\ell + 1}} 1 \qquad (\ell = 1, ..., r), \tag{5}$$

whence, by (4),

$$\begin{aligned} \sum_{d \mid (a, P)} \mu(d) \geqslant & \prod_{j=1}^{r} \Bigg(\sum_{\substack{d \mid (a, P_j) \\ \nu(d) \leqslant k_j}} \mu(d) \Bigg) \\ & - \sum_{\ell=1}^{r} \Bigg(\sum_{\substack{d \mid (a, P_\ell) \\ \nu(d) = k_\ell + 1}} 1 \Bigg) \prod_{\substack{j=1 \\ j \neq \ell}}^{r} \Bigg(\sum_{\substack{d \mid (a, P_j) \\ \nu(d) \leqslant k_j}} \mu(d) \Bigg) \end{aligned}$$

and therefore (cf. (3))

$$\begin{aligned} S(\mathscr{A}, \mathscr{P}) \geqslant & \sum_{\substack{d_1, ..., d_r \\ d_j \mid P_j, \, \nu(d_j) \leqslant k_j}} \mu(d_1) \cdots \mu(d_r) \, |\{a \in \mathscr{A} : d_1 \cdots d_r \mid a\}| \\ & - \sum_{\ell=1}^{r} \sum_{\substack{d_1, ..., d_r \\ d_j \mid P_j, \, \nu(d_j) \leqslant k_j (j \neq \ell) \\ d_\ell \mid P_\ell, \, \nu(d_\ell) = k_\ell + 1}} \mu\left(\frac{d_1 \cdots d_r}{d_\ell}\right) |\{a \in \mathscr{A} : d_1 \cdots d_r \mid a\}|. \end{aligned} \tag{6}$$

The proof of (5) is quite simple but, in any case, (5) will appear as a very special case of a certain general identity ([DHR, Lemma 2.1]) which we shall prove next.

2. A SIEVE IDENTITY

For each integer d let $p^-(d)$, $p^+(d)$ denote the least and largest prime factors respectively of d, and set $p^+(1) = 1$. Next, let $\chi(d)$ be any function defined on the set of all positive integer divisors d of P that has the following properties: (i) $\chi(1) = 1$, (ii) $\chi(d)$ assumes only the values 0 or 1; (iii) χ is divisor-closed in the sense that if $\chi(d) = 1$ and $t \mid d$ then $\chi(t) = 1$. Associate with χ its "complementary" function $\bar{\chi}(\cdot)$ given by

$$\bar{\chi}(1) = 0, \qquad \bar{\chi}(d) = \chi(d/p^-(d)) - \chi(d) \qquad \text{when} \quad d > 1, \quad d \mid P.$$

Note that $\bar{\chi}(d)$ also assumes only the values 0 or 1 and that $\bar{\chi}(d)=0$ when $\chi(d)=1$.

EXAMPLE. Let

$$\chi(d)=\chi^{(k)}(d)=\begin{cases}1, & \nu(d)\leqslant k,\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$\bar{\chi}^{(k)}(d)=1 \qquad \text{if and only if} \quad \nu(d)=k+1.$$

The identity we mentioned earlier first occurs in [HR, Chapter 2, Section 1], and is sometimes referred to as the "fundamental sieve identity"; it asserts that

LEMMA 2. *For any divisor D of P and any arithmetic function $h(\cdot)$,*

$$\sum_{d\mid D} h(d)=\sum_{d\mid D} h(d)\,\chi(d)+\sum_{d\mid D}\bar{\chi}(d)\sum_{\substack{t\mid D\\ p^+(t)<p^-(d)}} h(dt) \tag{7}$$

(*note that, in the second sum on the right, $d>1$ may be assumed since $\bar{\chi}(1)=0$*). *In particular, if h is multiplicative,*

$$\sum_{d\mid D} h(d)=\sum_{d\mid D} h(d)\,\chi(d)+\sum_{d\mid D} h(d)\,\bar{\chi}(d)\prod_{\substack{p\mid D\\ p<p^-(d)}}(1+h(p)). \tag{8}$$

Before we prove the identity we shall illustrate it by taking $h=\mu$. Since

$$\prod_{\substack{p\mid D\\ p<p^-(d)}}(1+\mu(p))=\begin{cases}1, & p^-(d)=p^-(D),\\ 0 & \text{otherwise,}\end{cases}$$

we obtain

$$\sum_{d\mid D}\mu(d)=\sum_{d\mid D}\mu(d)\,\chi(d)+\sum_{\substack{d\mid D\\ p^-(d)=p^-(D)}}\mu(d)\,\bar{\chi}(d), \tag{9}$$

and it follows in particular from the above example that

$$\sum_{d\mid D}\mu(d)=\sum_{\substack{d\mid D\\ \nu(d)\leqslant k}}\mu(d)+(-1)^{k+1}\sum_{\substack{d\mid D\\ p^-(d)=p^-(D)\\ \nu(d)=k+1}} 1,$$

so that (1) and (5) follow.

Proof of the Identity (from [DHR]). Suppose $d>1$ is any divisor of D, and write

$$d=p_1\cdots p_m, \qquad p_1>p_2>\cdots>p_m.$$

Then

$$1-\chi(d)=\sum_{i=1}^{m}(\chi(p_1\cdots p_{i-1})-\chi(p_1\cdots p_i))=\sum_{i=1}^{m}\bar{\chi}(p_1\cdots p_i)$$
$$=\sum_{\substack{\delta\mid d,\,\delta>1\\ p^+(d/\delta)<p^-(\delta)}}\bar{\chi}(\delta),$$

and therefore

$$\sum_{d\mid D}h(d)(1-\chi(d))=\sum_{\substack{d\mid D\\ d>1}}h(d)\sum_{\substack{\delta\mid d,\,\delta>1\\ p^+(d/\delta)<p^-(\delta)}}\bar{\chi}(\delta)$$
$$=\sum_{\delta\mid D,\,\delta>1}\bar{\chi}(\delta)\sum_{\substack{\delta t\mid D\\ p^+(t)<p^-(\delta)}}h(\delta t)$$
$$=\sum_{\delta\mid D,\,\delta>1}\bar{\chi}(\delta)\sum_{\substack{t\mid D\\ p^+(t)<p^-(\delta)}}h(\delta t).$$

This proves (7), and for multiplicative h (8) is obvious. ∎

3. THE MAIN RESULT

To progress beyond (3) and (6) we postulate some information about $|\{a\in\mathscr{A}: d\mid a\}|$ when $d\mid P$; and it is usual to assume that there exists a non-negative multiplicative arithmetic function $\omega(\cdot)$ such that the numbers

$$r_d:=|\{a\in\mathscr{A}: d\mid a\}|-\frac{\omega(d)}{d}X$$

are in some sense remainders (note that $r_1=|\mathscr{A}|-X$). Then, by (3),

$$S(\mathscr{A},\mathscr{P})\leqslant \Pi X+R \tag{10}$$

where

$$\Pi:=\prod_{j=1}^{r}\Bigg(\sum_{\substack{d\mid P_j\\ \nu(d)\leqslant k_j}}\mu(d)\frac{\omega(d)}{d}\Bigg)\quad\text{and}\quad R:=\sum_{\substack{d_1,\,\ldots,\,d_r\\ d_j\mid P_j,\,\nu(d_j)\leqslant k_j}}|r_{d_1\cdots d_r}|; \tag{11}$$

and similarly (6) leads to

$$S(\mathscr{A}, \mathscr{P}) \geqslant \Pi \left\{1 - \sum_{\ell=1}^{r} \left(\sum_{\substack{d \mid P_\ell \\ \nu(d) = k_\ell + 1}} \frac{\omega(d)}{d} \right) U_\ell^{-1} \right\} X - R - R' \tag{12}$$

where

$$U_\ell := \sum_{\substack{d \mid P_\ell \\ \nu(d) \leqslant k_\ell}} \mu(d) \frac{\omega(d)}{d} \qquad (\ell = 1, ..., r) \tag{13}$$

and

$$R' := \sum_{\ell=1}^{r} \sum_{\substack{d_1, ..., d_r \\ \nu(d_j) \leqslant k_j (j \neq \ell) \\ \nu(d_\ell) = k_\ell + 1}} |r_{d_1 \cdots d_r}|. \tag{14}$$

Write

$$W_j = \sum_{d \mid P_j} \mu(d) \frac{\omega(d)}{d} = \prod_{p \in \mathscr{P}_j} \left(1 - \frac{\omega(p)}{p}\right)$$

and

$$W = \sum_{d \mid P} \mu(d) \frac{\omega(d)}{d} = \prod_{p \in \mathscr{P}} \left(1 - \frac{\omega(p)}{p}\right) = W_1 W_2 \cdots W_r.$$

We expect $S(\mathscr{A}, \mathscr{P})$ to be comparable (in some sense) with XW. Apply (8) with $D = P_j$, $h(d) = \mu(d)\, \omega(d)/d$ and $\chi = \chi^{(k_j)}$ to deduce that

$$\begin{aligned} W_j = & \sum_{\substack{d \mid P_j \\ \nu(d) \leqslant k_j}} \mu(d) \frac{\omega(d)}{d} \\ & + (-1)^{k_j + 1} \sum_{\substack{d \mid P_j \\ \nu(d) = k_j + 1}} \frac{\omega(d)}{d} \prod_{\substack{p \in \mathscr{P}_j \\ p < p^-(d)}} \left(1 - \frac{\omega(p)}{p}\right), \end{aligned}$$

whence, for each $j = 1, ..., r$, since each k_j is even, we have

$$U_j - \sum_{\substack{d \mid P_j \\ \nu(d) = k_j + 1}} \frac{\omega(d)}{d} \leqslant W_j \leqslant U_j - W_j \sum_{\substack{d \mid P_j \\ \nu(d) = k_j + 1}} \frac{\omega(d)}{d}. \tag{15}$$

Also

$$\sum_{\substack{d \mid P_j \\ \nu(d)=k_j+1}} \frac{\omega(d)}{d} \leqslant \frac{1}{(k_j+1)!}\left(\sum_{p \in \mathscr{P}_j} \frac{\omega(p)}{p}\right)^{k_j+1}, \tag{16}$$

and

$$\sum_{p \in \mathscr{P}_j} \frac{\omega(p)}{p} \leqslant \sum_{p \in \mathscr{P}_j} \log\left(1-\frac{\omega(p)}{p}\right)^{-1} = \log W_j^{-1} =: L_j, \tag{17}$$

say. Hence, by (11), (15) and (16),

$$W \leqslant \Pi \leqslant W \prod_{j=1}^{r} e^{L_j}\left(1+\frac{L_j^{k_j+1}}{(k_j+1)!}\right) \leqslant W \exp E \tag{18}$$

on writing

$$E := \sum_{j=1}^{r} \frac{L_j^{k_j+1}}{(k_j+1)!} e^{L_j}; \tag{19}$$

and by (11) it follows that

$$S(\mathscr{A}, \mathscr{P}) \leqslant XW \exp E + R. \tag{20}$$

Next we turn to (12). By (15),

$$U_\ell^{-1} \leqslant W_\ell^{-1}(1+V_\ell)^{-1}, \qquad V_\ell := \sum_{\substack{d \mid P_\ell \\ \nu(d)=k_\ell+1}} \frac{\omega(d)}{d},$$

so that, using (17) and (18),

$$\begin{aligned} S(\mathscr{A}, \mathscr{P}) &\geqslant \{1-E'\}\ X\Pi - R - R' \\ &\geqslant \{1-E'\}\ XW - R - R' \end{aligned} \tag{21}$$

where

$$E' := \sum_{j=1}^{r} \frac{e^{L_j}}{1+L_j^{-1-k_j}(k_j+1)!}. \tag{22}$$

Since $E' < E$ we obtain the less precise but simpler bound

$$S(\mathscr{A}, \mathscr{P}) \geqslant \{1-E\}\ XW - R - R'. \tag{23}$$

To sum up:

THEOREM. *With E, E', R and R' as defined in* (19), (22), (11) *and* (14), *respectively, we have*

$$S(\mathscr{A}, \mathscr{P}) \leqslant X \prod_{p \in \mathscr{P}} \left(1 - \frac{\omega(p)}{p}\right) \exp E + R$$

and

$$S(\mathscr{A}, \mathscr{P}) \geqslant (1 - E')\, X \prod_{p \in \mathscr{P}} \left(1 - \frac{\omega(p)}{p}\right) - R - R'$$

$$\geqslant (1 - E)\, X \prod_{p \in \mathscr{P}} \left(1 - \frac{\omega(p)}{p}\right) - R - R'$$

From now on take $\mathscr{P}$ to be a set of primes in the interval $[2, z)$ and for each $j = 1, 2, ..., r$ let $\mathscr{P}_j = \mathscr{P} \cap [z_{j+1}, z_j)$ where

$$2 = z_{r+1} < z_r < \cdots < z_1 = z.$$

For the moment we also assume, as is often the case, that

$$|r_d| \leqslant \omega(d), \qquad d \mid P. \tag{23}$$

Then

$$\sum_{\substack{d \mid P_j \\ \nu(d) \leqslant k_j}} \omega(d) < z_j^{k_j} \sum_{d \mid P_j} \omega(d)/d = z_j^{k_j} \prod_{p \in \mathscr{P}_j} \left(1 + \frac{\omega(p)}{p}\right) \leqslant z_j^{k_j} W_j^{-1}$$

and hence, by (11),

$$R < \left(\prod_{j=1}^{r} z_j^{k_j}\right) W^{-1}.$$

Similarly,

$$R' < \left(\prod_{j=1}^{r} z_j^{k_j}\right) W^{-1} \sum_{\ell=1}^{r} z_\ell W_\ell V_\ell < z \left(\prod_{j=1}^{r} z_j^{k_j}\right) W^{-1} \sum_{\ell=1}^{r} \frac{L^{k_\ell+1}}{(k_\ell+1)!}$$

$$< z \left(\prod_{j=1}^{r} z_j^{k_j}\right) W^{-1} E$$

by (16), (17) and (19). We conclude that

COROLLARY. *With a partition of $\mathscr{P}$ of the kind described above, and assuming only the condition* (23), *we have*

$$S(\mathscr{A}, \mathscr{P}) \leqslant XW\{\exp E + \eta\}, \tag{24}$$

where

$$\eta = \left(\prod_{j=1}^{r} z_j^{k_j}\right) X^{-1} W^{-2};$$

and

$$S(\mathscr{A}, \mathscr{P}) \geqslant XW\{1 - E' - \eta - \eta z E\}. \tag{25}$$

We also consider another type of bound on the remainders r_d, by supposing that $|\mathscr{A}| = \pi(Y)$, the number of primes $\leqslant Y$, and for each $d \mid P$, there are $s(d)$ numbers $t_1, ..., t_{s(d)}$ so that

$$|\{a \in \mathscr{A} : d \mid a\}| = \sum_{h=1}^{s(d)} \pi(Y; d, t_h),$$

where $\pi(Y; d, t)$ is the number of primes $\leqslant Y$ in the residue class t mod d. Here $\omega(d) = ds(d)/\phi(d)$ (in particular $s(d)$ must be multiplicative) and

$$|r_d| \leqslant \sum_{h=1}^{s(d)} \left| \pi(Y; d, t_h) - \frac{\pi(Y)}{\phi(d)} \right|.$$

The quantities R and R' are then bounded using the Bombieri–A. I. Vinogradov Theorem. For every $B > 0$ there is a number A so that the following holds. If

$$\prod_{j=1}^{r} z_j^{k_j} \leqslant Y^{1/2} (\log Y)^{-A},$$

then $R \ll Y(\log Y)^{-B}$ and thus

$$S(\mathscr{A}, \mathscr{P}) \leqslant XW \exp E + O(Y(\log Y)^{-B}), \tag{26}$$

and if

$$z \prod_{j=1}^{r} z_j^{k_j} \leqslant Y^{1/2} (\log Y)^{-A},$$

then $R+R' \ll Y(\log Y)^{-B}$ and

$$S(\mathscr{A}, \mathscr{P}) \geqslant XW(1-E') - O(Y(\log Y)^{-B}). \tag{27}$$

For an appropriate choice of B, R and R' will be of smaller order than XW.

Remark. Michael Filaseta has pointed out to us that the Brun–Hooley sieve in the above form may also be applied to a more general type of sieve. If $\mathscr{A}$ is any finite set we may associate with each prime $p \in \mathscr{P}$ a subset $\mathscr{A}_p$. All of the above inequalities hold if we replace the quantity (a, P) by

$$\prod_{a \in \mathscr{A}_p} p$$

throughout.

4. APPLICATIONS

Inequalities (24)–(27) yield three kinds of results. We will concentrate on (24) and (25) for now, as the same type of bounds also follows from (26) and (27) in a similar fashion.

I. By (24),

$$S(\mathscr{A}, \mathscr{P}) \ll XW$$

provided only that E and η are bounded. This estimate has numerous applications as an auxiliary counting device.

II. Inequality (25) is non-trivial only if

$$E' + \eta + \eta z E < 1,$$

for example, if $E' < 1$ and $\eta z E = o(1)$ as $X \to \infty$. Then

$$S(\mathscr{A}, \mathscr{P}) > 0$$

tells us that there exists an element a of $\mathscr{A}$ all of whose prime factors from $\mathscr{P}$ are large; and if $\mathscr{P}$ is carefully chosen it will follow that a has very few prime factors in all. We shall give illustrations below.

III. Together, (24) and (25) yield

$$S(\mathscr{A}, \mathscr{P}) \sim XW \qquad \text{as} \quad X \to \infty,$$

provided that $z\eta$ is bounded and $E = o(1)$ as $X \to \infty$. This is a result of "fundamental lemma" type, and also has numerous applications.

We make all this clearer by choosing the sub-division points z_j and postulating some further information about the function ω. Let

$$z_r = \log\log X =: \xi \tag{28}$$

for short and

$$\log z_j = K^{1-j} \log z \qquad (j = 1, ..., r-1) \tag{29}$$

where $K > 1$ is a constant to be chosen conveniently. Of course we regard X as very large, and we determine r uniquely by

$$z^{K^{1-r}} \leqslant \xi < z_{r-1} = z^{K^{2-r}},$$

so that, in particular,

$$\frac{1}{\log \xi} \leqslant \frac{K^{r-1}}{\log z} < \frac{K}{\log \xi}. \tag{30}$$

We defer the choice of the even integers k_j except that we put $k_r = \infty$ always. This is in order provided we estimate the magnitude of a divisor d of P_r by $d < \xi^{\pi(\xi)} < \xi^{\xi}$ in place of ξ^{k_r}. As a consequence we have to modify the definition of η to

$$\eta = \left(\prod_{j=1}^{r-1} z_j^{k_j} \right) \xi^{\xi} X^{-1} W^{-2}, \tag{31}$$

and also note that, in the definitions (19) and (22) of E and E', the summation over j now runs from 1 to $r-1$ only.

Next we impose on $\omega(\cdot)$ the well-known Iwaniec condition:

(Ω) *Suppose there exist positive constants κ and A such that*

$$\prod_{y_1 \leqslant p < y_2} \left(1 - \frac{\omega(p)}{p}\right)^{-1} \leqslant \left(\frac{\log y_2}{\log y_1}\right)^{\kappa} \exp\left(\frac{A}{\log y_1}\right), \qquad 2 \leqslant y_1 < y_2.$$

Then

$$W^{-1} \leqslant \left(\frac{\log z}{\log 2}\right)^{\kappa} \exp\left(\frac{A}{\log 2}\right) = B(\log z)^{\kappa}, \quad B = \frac{\exp(A/\log 2)}{(\log 2)^{\kappa}}, \tag{32}$$

and, by (17),

$$L_j \leqslant \kappa \log\left(\frac{\log z_j}{\log z_{j+1}}\right) + \frac{A}{\log z_{j+1}} = \kappa \log K + \frac{AK^j}{\log z} \qquad (1 \leqslant j \leqslant r-1),$$

so that, by (30),

$$L_j < \kappa \log K + \frac{AK}{\log \xi} =: L \qquad (1 \leqslant j \leqslant r-1), \tag{33}$$

say.

Let us write

$$z = X^{1/u}, \qquad u > 1;$$

then, by (31),

$$\eta \leqslant B^2 X^{(\Gamma/u)-1} (\log X)^{2\kappa + \log \xi}, \qquad \Gamma := \sum_{j=1}^{r-1} \frac{k_j}{K^{j-1}}. \tag{34}$$

Also, by (19)

$$E < e^L \sum_{j=1}^{r-1} \frac{L^{k_j+1}}{(k_j+1)!} \tag{35}$$

and by (22)

$$E' < e^L \sum_{j=1}^{r-1} \frac{1}{1 + L^{-k_j-1}(k_j+1)!}. \tag{36}$$

We see from (34) that

$$z\eta = o(1) \qquad \text{as} \quad X \to \infty \quad \text{if } \ \Gamma + 1 < u. \tag{37}$$

Choosing the even integers $k_1, ..., k_{r-1}$ depends on the kind of application one has in mind. In categories I and III a reasonable all-purpose choice is

$$k_j = b + 2(j-1), \qquad j = 1, ..., r-1,$$

where $b \geqslant 2$ is an even integer that remains at our disposal. Here

$$\Gamma = \sum_{i=0}^{r-2} \frac{b+2i}{K^i} < \frac{bK}{K-1} + \frac{2K}{(K-1)^2},$$

so that $z\eta = o(1)$ if

$$u > 1 + \frac{bK^2 - (b-2)\,K}{(K-1)^2};$$

also, by (35) (and bearing in mind an earlier remark)

$$E \leqslant e^L \sum_{j=1}^{r-1} \frac{L^{b+1+2(j-1)}}{(b+1+2(j-1))!} = e^L \sum_{i=0}^{r-2} \frac{L^{b+1+2i}}{(b+1+2i)!}$$
$$\leqslant \frac{L^{b+1}}{(b+1)!} e^L \sum_{i=0}^{\infty} \frac{L^{2i}}{(2i)!} < \frac{L^{b+1}}{(b+1)!} e^{2L} < \left(\frac{eL}{b+1}\right)^{b+1} e^{2L}.$$

By (33), $L < 1.01\kappa \log K$ if x is large enough. Taking $K = e^{150/101}$ and $b = 2$ we see that $z\eta = o(1)$ as $X \to \infty$ if $u > 4.35$, and that

$$E < (\tfrac{1}{2}\kappa e^{1+\kappa})^3.$$

This suffices for applications of type I.

For applications of type III we choose b large. For example, take $K = 2 + \sqrt{2}$ and

$$b = 2(\lfloor \xi \rfloor + 1) > 2\xi,$$

so that $z\eta = o(1)$ if $u > 4\xi$ and

$$E < \left(\frac{1.69\kappa}{\xi}\right)^{2\xi} e^{2.49\kappa} \to 0 \qquad \text{as} \quad X \to \infty.$$

Notice that here we sieve only up to $z = X^{1/(4 \log\log X)}$, but obtain asymptotic equality for $S(\mathscr{A}, \mathscr{P})$.

For applications of type II we have to proceed more carefully in order to arrive at the best results of which the method is capable (subject to (29)). Specifically, we have to choose $k_1, ..., k_{r-1}$ and K so as to *minimize*

$$1 + \Gamma = 1 + \sum_{j=1}^{r-1} \frac{k_j}{K^{j-1}} \tag{38}$$

subject to

$$e^L \sum_{j=1}^{r-1} \frac{1}{1 + (k_j+1)!\, L^{-1-k_j}} < 1. \tag{39}$$

The best procedure in this optimization exercise is, given a candidate K, to take as many k_j as possible to be 2 (as many as (39) allows), then take as many as possible to be 4, etc. By (33), it is in order to take $L = \kappa \log K$ for purposes of numerical computation, so that $e^L = K^\kappa$. With a candidate K and

$$b(k) := \frac{K^\kappa}{1 + (k+1)!\, (\kappa \log K)^{-k-1}},$$

the explicit procedure is to take the first $n_2 = \lfloor 1/b(2) \rfloor$ k_j's to be 2, the next $n_4 = \lfloor (1 - n_2 b(2))/b(4) \rfloor$ k_j's to be 4, etc. In this way (35) remains true automatically while the candidate K in conjunction with n_2 twos, n_4 fours, etc. determines $1+\Gamma$.

The following example will serve as an illustration.

EXAMPLE. Let $\mathscr{A} = \{n^2+1 : n \leqslant x\}$ and $\mathscr{P} = \{2\} \cup \{p < z : \quad p \equiv 1 \bmod 4\}$. Here $X = x$, $\omega(2) = 1$, $\omega(p) = 2$ when $p \equiv 1 \bmod 4$ $(\omega(p) = 0$ otherwise), and

$$\prod_{y_1 \leqslant p < y_2} \left(1 - \frac{\omega(p)}{p}\right)^{-1} = \prod_{\substack{y_1 \leqslant p < y_2 \\ p \equiv 1 \ (\mathrm{mod}\ 4)}} \left(1 - \frac{2}{p}\right)^{-1}, \qquad 2 < y_1 < y_2,$$

$$= \frac{\log y_2}{\log y_1} \left(1 + O\left(\frac{1}{\log y_1}\right)\right).$$

Thus the Iwaniec condition (Ω) holds with $\kappa = 1$.

The best choice of K turns out to be 2.57195, and one finds that $n_2 = 3$, $n_4 = 3$, $n_6 = 4$, $n_8 = 5$, etc., and therefore $1 + \Gamma < 4.4763$. Take u to be 4.48 and $z = x^{1/u} = x^{1/4.48}$. We may conclude that $\mathscr{A}$ contains $\gg x/\log x$ elements having no prime factor $< x^{1/4.48}$, and each of these elements obviously cannot have more than 8 prime factors, or, as we say, is a P_8.

The following table summarizes the best choices for $\kappa = 1$, 2, 3, 4, 5.

κ	K	u	k_1
1	2.57195	4.4763	2
2	1.54226	7.7428	2
3	1.33100	11.7172	2
4	1.42322	15.6527	4
5	1.31560	19.3627	4

The interested reader should be able to verify easy, using $\kappa = 2$, that the number of prime twins not exceeding x is $\ll x(\log x)^{-2}$, and that there exists infinitely many integers such that each of n, $n+2$ is the product of at most 7 prime factors. The much more complicated Brun's sieve gives nothing better.

Although dealing with a set $\mathscr{A}$ which is of the form $\{f(p)$: $p \leqslant X$, p prime$\}$, where f is a polynomial, requires an additional result (the Bombieri–A. I. Vinogradov Theorem), it is still straightforward to obtain bounds in this case. For Type II results, we note that (27) holds provided that $u > 2(\Gamma+1)$, where Γ is given by (38) and we require (39) to hold. For example, if $\mathscr{A} = \{p+2 : p \leqslant X, p \text{ prime}\}$, so that $\kappa = 1$, it follows that for infinitely many primes p, $p+2$ is composed of prime factors $\leqslant X^{1/8.96}$, which implies that $p+2 = P_8$.

We are indebted to the referee for several helpful remarks, and especially for pointing out that the remainder sums R and R' have, potentially, a highly flexible structure—for example, we could leave R in the form

$$\sum_{\substack{d_1, \ldots, d_r \\ d_j \mid P_j,\, \nu(d_j) \leqslant k_j}} \mu(d_1) \cdots \mu(d_r)\, r_{d_1 \cdots d_r}$$

—and that there are perhaps applications where this would be an advantage, for instance if one were then able to use more recent and sharper versions of the Bombieri–Vinogradov theorem. In the case of the prime twin conjecture, however, any such refinement if deployed above would not improve on what can be accomplished by the more sophisticated Rosser–Iwaniec sieve methods.

5. A DUAL OF HOOLEY'S METHOD

This method in the form of inequality (4) lends itself to a dual purpose. Rather than aim for full generality here, consider the case of

$$\mathscr{A} = \left\{ \prod_{j=1}^{r} (a_j n + b_j) : n \leqslant x \right\}, \qquad r \geqslant 2,$$

where the a_j, b_j are integers satisfying

$$\prod_{j=1}^{r} a_j \prod_{1 \leqslant i < j \leqslant r} (a_i b_j - a_j b_i) \neq 0,$$

and the polynomial

$$F(n) := \prod_{j=1}^{r} (a_j n + b_j)$$

has no fixed prime divisors. Let $\mathscr{P}$ be the set of all primes truncated at some z. Obviously we are here addressing a generalized prime k-tuples conjecture, and the problem of estimating $S(\mathscr{A}, \mathscr{P})$ is of "dimension" r, that is, has $\kappa = r$. However, following the "vector" sieve of Brüdern and Fouvry mentioned at the start, we have

$$\sum_{d \mid (F(n), P)} \mu(d) = \prod_{j=1}^{r} \sum_{d \mid (a_j n + b_j, P)} \mu(d) \leqslant \prod_{j=1}^{r} \sum_{d \mid (a_j n + b_j, P)} \mu(d) \chi^{+}(d)$$

where $\chi^{+}(d)$ characterizes the LINEAR upper Rosser–Iwaniec sieve; and, as in (4),

$$\sum_{d \mid (F(n), P)} \mu(d) \geqslant \prod_{j=1}^{r} \sum_{d \mid (a_j n + b_j, P)} \mu(d) \chi^{+}(d) - \sum_{\ell=1}^{r} \Bigg(\sum_{\substack{d \mid (a_\ell n + b_\ell, P) \\ p^{-}(d) = p^{-}((a_\ell n + b_\ell, P))}} \bar{\chi}^{+}(d) \Bigg) \times \prod_{\substack{j=1 \\ j \neq \ell}}^{r} \Bigg(\sum_{d \mid (a_j n + b_j, P)} \mu(d) \chi^{+}(d) \Bigg).$$

This seems to us superior to Lemma 13 of [BF1] or (2.6) of [BF2] in the treatment of the "$y_\ell - x_\ell$" terms, and might lead to better results.

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